BAULKHAM HILLS HIGH SCHOOL

2018

YEAR 12 TERM 2 ASSESSMENTS

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 60 minutes
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- Show all relevant mathematical reasoning and/or calculations
- Marks may be deducted for careless or badly arranged work
- A reference sheet is provided at the back of this paper

Total marks: 41

Attempt all Questions 1 – 4 (pages 2-5)

Question 1 (9 Marks) Start on the appropriate page of your answer booklet.

a) Find the value of the constant term in the expansion of $\left(3x^2+\frac{2}{x}\right)^{12}$. **2**

b) The motion of an object is defined by the equation $x=4\sin(\pi t)$ where x is the displacement from the origin in centimetres at time t seconds.

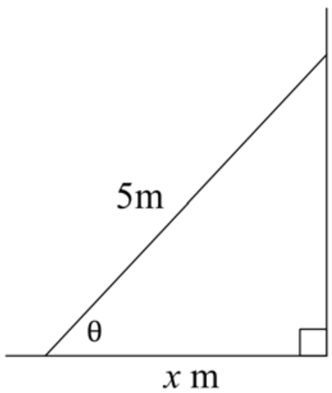
(i) Write down the amplitude of the motion. **1**

(ii) What is the period of the motion in seconds? **1**

(iii) What is the greatest speed of the object? **1**

(iv) Prove that the object is moving in Simple Harmonic Motion. **1**

c) A 5 metre ladder rests with one end against a wall and the other end on horizontal ground which is level with the base of the wall and x metres from the bottom of the wall. The end which is in contact with the ground is sliding away from the wall at a constant rate of 0.1 m/s.



Find the rate (in radians/s) at which the angle, θ, between the ladder and the ground is decreasing when the end of the ladder is 3 metres from the wall. **3**

End of Question 1

Question 2 (13 Marks) Start on the appropriate page of your answer booklet.

a) The acceleration of a particle undergoing Simple Harmonic Motion is given by $\ddot{x} = -n^2 x$, where x is the displacement of the particle from the origin in metres, and n is a positive constant. The particle starts from rest at a distance of 10 metres to the right of its centre of oscillation O. The period of the motion is 2 seconds.

(i) Prove that $v^2 = \pi^2(100 - x^2)$, where v is the velocity of the particle in metres per second. **3**

(ii) Find the speed of the particle in metres per second when it is 6 metres from the origin. **1**

(iii) Find the time taken by the particle to first reach the point 6 metres to the right of O, correct to 2 decimal places. (You may assume $x = a\cos nt$ where a is a constant is a solution of $\ddot{x} = -n^2 x$). **2**

b) The rate of growth of a population N over t years is given by: $\dfrac{dN}{dt} = -k(N - 700)$.

(i) Show that $N = 700 + Ae^{-kt}$ satisfies $\dfrac{dN}{dt} = -k(N - 700)$ where A and k are constants. **1**

(ii) The population decreased from an initial population of 8300 to 5100 in 5 years.
Find the population at the end of the next 5 years. Give your answer correct to the nearest hundred. **3**

c) Consider the binomial expansion $1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n$.

(i) Show that $1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$. **1**

(ii) Show that

$$\left[1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}\right]\left[1 + \frac{1}{2}\binom{n}{1} + \frac{1}{4}\binom{n}{2} + \ldots + \frac{1}{2^n}\binom{n}{n}\right] = 3^n.$$ **2**

End of Question 2

Question 3 (9 Marks) Start on the appropriate page of your answer booklet.

a) The coefficients of x^4 and x^5 in the expansion of $(2 + 5x)^n$ are in the ratio of 1:3 **2**

Find the value of *n*.

b) A certain particle moves along the *x* axis according to the equation $t = 2x^2 - 5x + 3$, where *x* is its displacement measured in centimetres at time *t* seconds.
Initially the particle is $1 \cdot 5$ cm to the right of the origin, *O*, and moving away from *O*.

(i) Prove that the velocity, v cm s^{-1}, is given by $v = \dfrac{1}{4x - 5}$. **2**

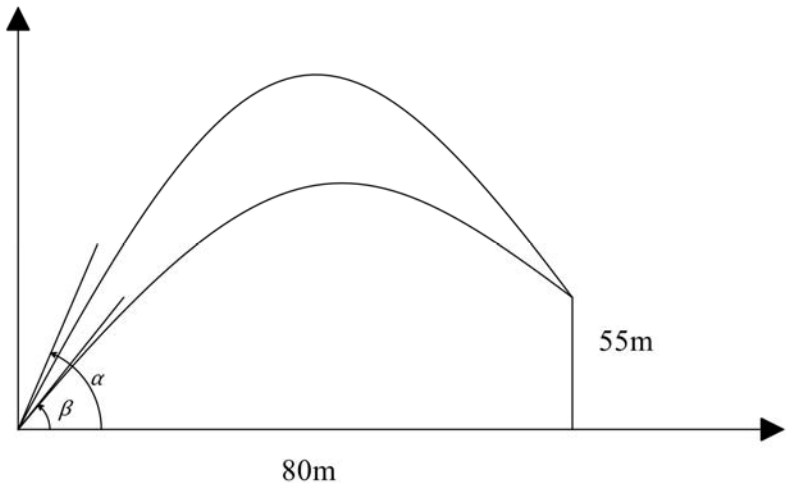
(ii) Find the velocity of the particle when $t = 6$ seconds. **3**

(iii) Find an expression for the acceleration, a cm s^{-2}, in terms of *x*. **2**

End of Question 3

Question 4 (10 Marks) Start on the appropriate page of your answer booklet.

a) A missile is fired from ground level into the air at a velocity of 40m/s and at an angle of α with the horizontal.



A short time later another missile is fired from the same point and with the same speed but at a different angle β. Both missiles hit the same target at the same time. The target is 55m above the ground and 80m horizontally from the point of firing. (Take g =10m/s^2 and neglect air resistance).

Given the equations of motion at time t seconds are:
$x = 40t\cos\alpha$ and $y = -5t^2 + 40t\sin\alpha$ (DO NOT PROVE)

(i) Show that the path of the first missile is given by $y = x\tan\alpha - x^2\left(\frac{\sec^2\alpha}{320}\right)$. **2**

(ii) Find the values of $\tan\alpha$ and $\tan\beta$. **2**

(iii) Determine the time difference between the firing times of the two missiles. **2**

b) Consider the expansion $(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \ldots + (-1)^n\binom{n}{n}x^n$.

(i) Show that $\int_0^1 (1-x)^n\,dx = \binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \ldots + \frac{(-1)^n}{n+1}\binom{n}{n}$. **2**

(ii) Hence find the value of $\sum_{r=0}^{2018}(-1)^r\frac{1}{r+1}\binom{2018}{r}$. **2**

End of Paper

HSC EXTENSION 1 TASK 3 JUNE 2018

Q1 a) General term: ${}^{12}C_k (3x^2)^{12-k} (2x^{-1})^k$ — ② correct answer

$= {}^{12}C_k 3^{12-k} 2^k x^{24-2k-k}$ — ① finds general term

$= {}^{12}C_k 3^{12-k} 2^k x^{24-3k}$ — NB unsimplified is acceptable

For constant term $24-3k=0$

$k=8$

$\therefore$ Constant term is ${}^{12}C_8 3^4 2^8$ $(=10264320)$

b (i) Amplitude $=4$ — ① correct answer

(ii) Period $= \frac{2\pi}{\pi}$ — ① correct answer

$=2$

(iii) $\dot{x} = 4\pi \cos(\pi t)$ — ① correct answer

$\therefore$ Greatest speed $= 4\pi$ cm/s.

(iv) $\ddot{x} = -4\pi^2 \sin(\pi t)$ — ① correct solution

$\ddot{x} = -\pi^2 x$ which is of the form $\ddot{x} = -n^2 x$.

$\therefore$ Object is moving in SHM.

c) $\cos\theta = \frac{x}{5}$

③ correct solution

② finds value of $\frac{d\theta}{dx}$ or $\frac{dx}{d\theta}$

① establishes correct chain rule

$\theta = \cos^{-1}\frac{x}{5}$

$\frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{-1}{\sqrt{25-x^2}} \times 0.1$

When $x=3$ $\frac{d\theta}{dt} = \frac{-1}{\sqrt{16}} \times \frac{1}{10}$

$= -\frac{1}{40}$ radians/sec.

2 a) (i) Period $= \frac{2\pi}{n}$

$$2 = \frac{2\pi}{n}$$
$$n = \pi$$
$$\ddot{x} = -\pi^2 x$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\pi^2 x$$
$$\left[\frac{1}{2}v^2\right]_0^v = -\pi^2\left[\frac{x^2}{2}\right]_{10}^x$$
$$\frac{1}{2}(v^2 - 0) = -\pi^2\left(\frac{x^2}{2} - \frac{10^2}{2}\right)$$
$$\frac{v^2}{2} = \pi^2\left(\frac{100 - x^2}{2}\right)$$
$$v^2 = \pi^2(100 - x^2)$$

(3) correct solution
(2) finds $\frac{1}{2}v^2$ in terms of x (including constant)
(1) finds the value of n
OR (1) attempts to integrate $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2 x$

(ii) When $x = 4$, $v^2 = \pi^2(100 - 36)$

$$v^2 = 64\pi^2$$
$$v = \pm \pi\sqrt{64}$$
$\therefore$ Speed $= 8\pi$ m/s

(1) correct answer

(iii) When $x = 6$, $6 = 10\cos \pi t$

$$\frac{3}{5} = \cos \pi t$$
$$\pi t = 0.92729\ldots$$
$$\therefore t = 0.295167\ldots$$
$t = 0.30$ seconds (2dp)

(2) correct answer
(1) substitutes $x = 6$ into $x = 10\cos \pi t$

2(b)(i) $N = 700 + Ae^{-kt}$ ① correct solution

$$\frac{dN}{dt} = -Ake^{-kt}$$
$$= -k(700 + Ae^{-kt} - 700)$$
$$= -k(N - 700)$$

ii) When $t=0$, $N = 8300$ ③ correct answer

② finds A and k

① finds correct value of A and attempts to find k

$$8300 = 700 + Ae^{0}$$
$$A = 7600$$
$$\therefore N = 700 + 7600e^{-kt}$$

When $t=5$, $N = 5100$
$$5100 = 700 + 7600e^{-5k}$$
$$4400 = 7600e^{-5k}$$
$$\frac{11}{19} = e^{-5k}$$
$$-5k = \ln\frac{11}{19}$$
$$5k = \ln\frac{19}{11}$$
$$k = \frac{\ln\frac{19}{11}}{5}$$
$$(\doteqdot 0.1093\ldots)$$

When $t = 10$ $N = 700 + 7600e^{-10\times\frac{\ln\frac{19}{11}}{5}}$
$$= 3247.368\ldots$$
$\therefore$ Population is approximately 3200

2c) i) Let $x = 1$ ① correct solution

$$1 + \binom{n}{1}1 + \binom{n}{2}1 + \ldots \binom{n}{n}1^n = (1+1)^n$$
$$1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$$

ii) Let $x = \frac{1}{2}$ in $1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n = (1+x)^n$ ② correct solution

$\therefore 1 + \frac{1}{2}\binom{n}{1} + \frac{1}{4}\binom{n}{2} + \ldots \frac{1}{2^n}\binom{n}{n} = (1 + \frac{1}{2})^n$ ① substitutes $x = \frac{1}{2}$

$$\left[1 + \binom{n}{1} + \binom{n}{2} + \ldots \binom{n}{n}\right]\left[1 + \frac{1}{2}\binom{n}{1} + \frac{1}{4}\binom{n}{2} + \ldots + \frac{1}{2^n}\binom{n}{n}\right] = 2^n \times 1.5^n = (2\times1.5)^n$$
$$= 3^n$$

3 (a) General term: ${}^nC_k\, 2^{n-k}(5x)^k$

$${}^nC_k\, 2^{n-k} 5^k x^k$$

$$\therefore \frac{{}^nC_4\, 2^{n-4} 5^4}{{}^nC_5\, 2^{n-5} 5^5} = \frac{1}{3}$$

$$\frac{2}{5}\frac{\frac{n!}{(n-4)!4!}}{\frac{n!}{(n-5)!5!}} = \frac{1}{3}$$

$$\frac{\cancel{(n-5)!} \times 8}{(n-4)\cancel{(n-5)!}} = \frac{8}{6}$$

$$n-4=6$$

$$n=10$$

(2) correct answer
(1) equates ratio of coefficients to 1/3.

b) i) $t = 2x^2-5x+3$

$$\frac{dt}{dx} = 4x-5$$

$$\frac{dx}{dt} = \frac{1}{4x-5}$$

(2) correct solution
(1) finds $\frac{dt}{dx}$

(ii) When $t=6$, $6 = 2x^2-5x+3$

$$2x^2-5x-3=0$$

$$(2x+1)(x-3)=0$$

$$x=-\tfrac{1}{2},\ x=3.$$

But when $t=0$, $x=1.5$ and $v=\frac{1}{1}=1$ cm/s.
Since $v \neq 0$, particle never stops and can't reach $-\frac{1}{2}$.

$$\therefore v = \frac{1}{4\times3-5} = \frac{1}{7}\text{ cm/s.}$$

(3) correct solution
(2) finds $v = \pm\frac{1}{7}$
(1) calculates $x=3$

iii) $a = \frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

$$= \frac{1}{2}\frac{d}{dx}(4x-5)^{-2}$$

$$= \tfrac{1}{2}\times -2\times 4\,(4x-5)^{-3}$$

$$a = \frac{-4}{(4x-5)^3}$$

(2) correct solution
(1) attempts to differentiate $\frac{d}{dx}\left(\tfrac{1}{2}v^2\right)$

4 a)(i) From $x = 40t\cos\alpha$

$$t = \frac{x}{40\cos\alpha}$$

sub in $y = -5t^2 + 40t\sin\alpha$

$$y = -5\frac{x^2}{1600\cos^2\alpha} + \cancel{40}\frac{x}{\cancel{40}\cos\alpha}\sin\alpha$$

$$y = x\tan\alpha - \frac{x^2}{320}\sec^2\alpha$$

(2) correct solution
(1) correctly attempts to eliminate t from displacement equations

(ii) When $x = 80$ $y = 55$

$$55 = 80\tan\alpha - \frac{6400\sec^2\alpha}{320}$$

$$55 = 80\tan\alpha - 20(\tan^2\alpha + 1)$$

$$20\tan^2\alpha - 80\tan\alpha + 75 = 0$$

$$4\tan^2\alpha - 16\tan\alpha + 15 = 0$$

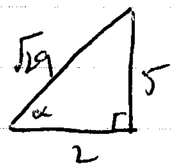
$$(2\tan\alpha - 3)(2\tan\alpha - 5) = 0$$

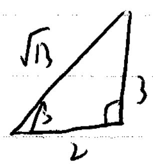
$$\tan\alpha = \tfrac{3}{2}, \quad \tan\alpha = \frac{5}{2}$$

$$\therefore \tan\alpha = \frac{5}{2} \quad \& \quad \tan\beta = \frac{3}{2}$$

(2) correct solution (must specify $\tan\alpha = \frac{5}{2}$ and $\tan\beta = \frac{3}{2}$ as $(\alpha > \beta)$)
(1) quadratic in $\tan\alpha$

iii)





$x = 40t\cos\alpha$ $\qquad$ $x = 40t\cos\beta$

$80 = 40t\frac{2}{\sqrt{29}}$ $\qquad$ $80 = 40t\frac{2}{\sqrt{13}}$

$t = \sqrt{29}$ $\qquad$ $t = \sqrt{13}$

$\therefore$ Difference in time $= \sqrt{29} - \sqrt{13}$

$= 1.7796\ldots$ secs

(2) correct answer
(1) calculates time for one of the missiles

4 iii) When $x = 80, y = 55$

$$55 = 80\tan\alpha - \frac{6400 \sec^2\alpha}{320}$$

$$55 = 80\tan\alpha - 20(\tan^2\alpha + 1)$$

$$20\tan^2\alpha - 80\tan\alpha + 75 = 0$$

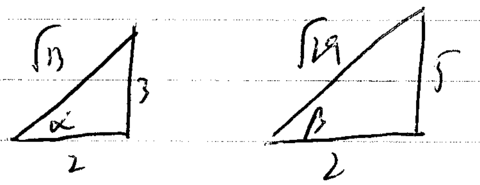
$$4\tan^2\alpha - 16\tan\alpha + 15 = 0$$

$$(2\tan\alpha - 3)(2\tan\alpha - 5) = 0$$

$\tan\alpha = 3/2$ $\quad \tan\alpha = 5/2$

$\therefore \tan\alpha = 3/2$ & $\tan\beta = 5/2$

(2) correct solution
(1) solves quadratic in $\tan\alpha$.

(iv) 

$x = 40t\cos\alpha$

$80 = 40t\,\frac{2}{\sqrt{13}}$

$t = \sqrt{13}$

$x = 40t\cos\beta$

$80 = 40t\,\frac{2}{\sqrt{29}}$

$t = \sqrt{29}$

$\therefore$ difference $= \sqrt{29} - \sqrt{13}$

$= 1.7796\ldots$ secs

(2) correct answer
(1) calculates α
(1) calculates time for 1 missile.

4 b) i) $\int_0^1 (1-x)^n dx = \int_0^1 \left(\binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 + \dots + (-1)^n \binom{n}{n} x^n\right) dx$

$= \left[\binom{n}{0}x - \binom{n}{1}\frac{x^2}{2} + \binom{n}{2}\frac{x^3}{3} + \dots + (-1)^n \binom{n}{n}\frac{x^{n+1}}{n+1}\right]_0^1$

$= \left(\binom{n}{0} - \binom{n}{1}\frac{1}{2} + \binom{n}{2}\frac{1}{3} + \dots + (-1)^n\binom{n}{n}\frac{1}{n+1}\right) - \left(0 - 0 + 0 + \dots 0\right)$

$= \binom{n}{0} - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \dots + \frac{(-1)^n}{n+1}\binom{n}{n}$

(2) Correct solution
(1) Integrates RHS

b (ii) $\int_0^1 (1-x)^{2018} dx = \left[\frac{-(1-x)^{2019}}{2019}\right]_0^1$

$= 0 + \frac{1^{2019}}{2019}$

$= \frac{1}{2019}$

(2) correct solution
(1) integrates $\int_0^1 (1-x)^n dx$

$\therefore$ From (i) $\binom{2018}{0} - \frac{1}{2}\binom{2018}{1} + \frac{1}{3}\binom{2018}{2} + \dots + \frac{(-1)^{2018}}{2019}\binom{2018}{2018} = \sum_{r=0}^{2018} (-1)^r \frac{1}{r+1}\binom{2018}{r}$

$\therefore \sum_{r=0}^{2018} (-1)^r \frac{1}{r+1}\binom{2018}{r} = \frac{1}{2019}$